Generalised structure of manifold

II International Workshop on Geometric Field Theory

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Universidad Nacional de Educación a Distancia

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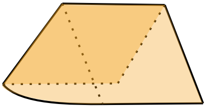

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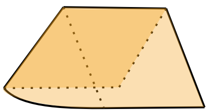 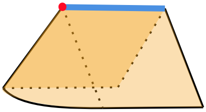

Singular foliation

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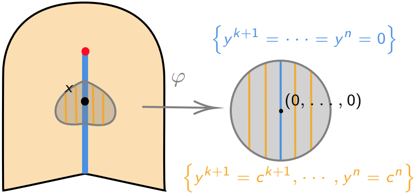
> For each $x \in M$, if $\dim(\mathcal{F}(x)) = k$, there is a chart of M in a neighbourhood U of x such that $\{y^{k+1} = \cdots = y^n = 0\}$ is the path-connected component of $\mathcal{F}(x) \cap U$ that contains x, and each $\{y^{k+1} = c^{k+1}, \ldots, y^n = c^n\}$ it wholly contained in some leaf of $\mathcal{F}$



Integrability

We are going to find this foliation by finding its tangent distribution. But, When is a distribution the tangent distribution of a foliation? When is it integrable?

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With regular foliations:

Frobenious Theorem

A regular distribution D is integrable if and only if it is *involutive*:

$$[X, Y] \in D \ \forall X, Y \in D,$$

Stefan-Sussman

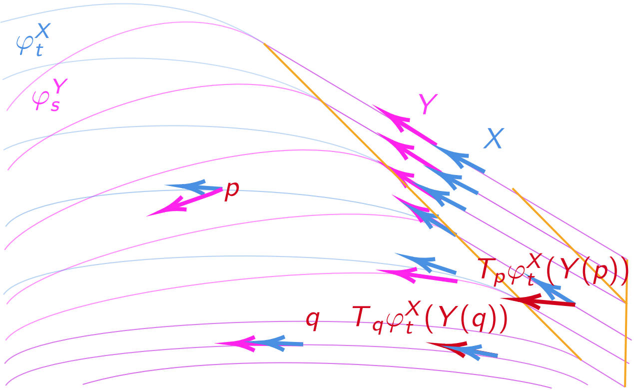
For singular foliations, we have:

Stefan-Sussman Theorem

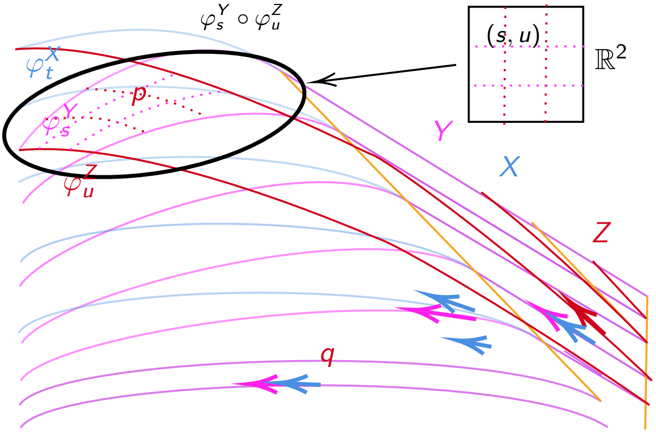
A singular distribution is integrable if and only if it is generated by a family C of vector fields and:

$$T_x\varphi_t^X(D_x) = D_x \ \forall X \in C$$

It is indeed a necessary condition, since it implies being involutive:



And it is sufficient, because we can construct a foliated chart with the flows of C:



Groupoids

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For example, the composition of invertible morphisms:

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In this example we have plenty of maps:

1. The "source" α and the "target" β

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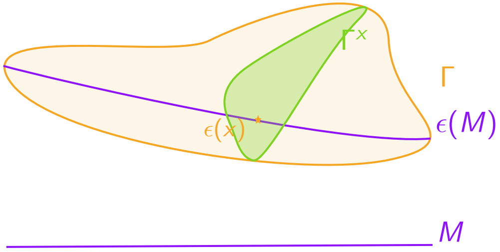
And there must exist a set were α and β go, and where we want the identities to exist. Let's call it M.

Definition

A Lie groupoid is a groupoid $\Gamma \rightrightarrows M$ where Γ and M are smooth manifolds, the previous mentioned maps are smooth and the source and the target are submersions.



Characteristic foliation

Groupoids

Given a subgoupoid $\overline{\Gamma}$ of Γ, is the distribution generated by the local vector fields $X \in \mathfrak{X}_{loc}(\Gamma)$ that are:

(i) Invariant by left translations.

(ii) The flow φ_t^X of X at $\bar{g}$ is contained in $\overline{\Gamma}$, $\forall \bar{g} \in \overline{\Gamma}$.

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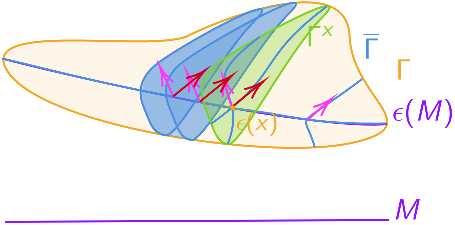

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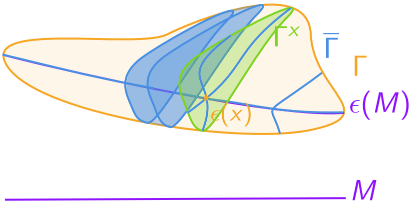
The characteristic distribution is invariant by this family of vector fields that generate it, so is integrable.



Characteristic foliation

Theorem 1[1]

Given a subgroupoid $\overline{\Gamma}$ of the Lie groupoid $\Gamma \rightrightarrows M$, there exists a foliation $\overline{\mathcal{F}}$ of Γ such that $\overline{\Gamma}$ is a union of leaves of $\overline{\mathcal{F}}$

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Theorem 2[2]

There is also a foliation $\mathcal{F}$ of M, called the base-characteristic foliation of $\overline{\Gamma}$ such that for each $x \in M$ there exists a transitive Lie subgroupoid $\overline{\Gamma}(\mathcal{F}(x))$ of Γ with base $\mathcal{F}(x)$.

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And these foliations are maximal.

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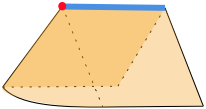
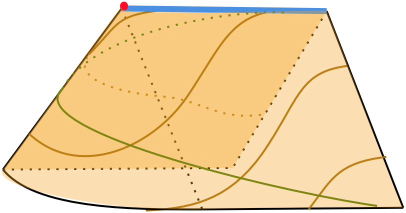


Application

In this case, the tangent distribution is generated by the vector fields $X \in \mathfrak{X}_{loc}(M)$ such that:

$$\varphi_t^X(x) \in N \ \forall x \in N$$



Further questions

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- Can we assign to N a diffeological structure so that it has constant dimension?

Thank you for your attention